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On periodic solutions of fractional-order differential systems with a fixed length of sliding memory

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Abstract. The fractional-order derivative of a non-constant periodic function is not periodic with the same period. Consequently, any time-invariant fractional-order systems do not have a non-constant periodic solution. This property limits the applicability of fractional derivatives and makes it unfavorable to model periodic real phenomena. This article introduces a modification to the Caputo and Rieman-Liouville fractional-order operators by fixing their memory length and varying the lower terminal. It is shown that this modified definition of fractional derivative preserves the periodicity. Therefore, periodic solutions can be expected in fractional-order systems in terms of the new fractional derivative operator. To confirm this assertion, one investigates two examples, one linear system for which one gives an exact periodic solution by its analytical expression and another nonlinear system for which one provides exact periodic solutions using qualitative and numerical methods.

Keywords: Fractional-order derivative; sliding fixed memory length; periodic solution.

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1 Introduction

The history of fractional calculus goes back to the end of the 17th century when L'Hopital asked Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction? Since that, time-fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent [18]. The advantages of fractional calculus have been described and pointed out in the last few decades by many authors [8, 15–19]. It has been shown that the fractional-order models of realistic systems are regularly more adequate than usually used integer-order models. Applications of these fractional-order models spread in many fields, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, quantum evolution of complex systems, and so on [6, 10, 11, 14, 20]. There are three definitions most frequently used

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for the general fractional differential operators. The first one is the Grünwald-Letnikov (GL) fractional differential operator defined by the limit of a fractional-order backward difference and has an advantage for numerical simulations. The second type is the Riemann-Liouville (RL) definition; this operator played a pivotal role in developing the fractional calculus theory. Using these two fractional differential operators in modeling real phenomena leads to mathematical models with initial conditions expressed in terms of fractional derivatives that do not have known physical interpretation. The third type is the Caputo derivative having the advantage of dealing models with initial conditions expressed in terms of the field variables and their integer-order derivatives, having clear physical interpretations [9]. Recently it has been demonstrated that the fractional-order derivative of a non-constant periodic function is not a periodic function with the same period [13,22,23] and in [5] the authors studied quasi-periodic properties of fractional order integrals and derivatives of periodic functions. As a consequence of the non-periodicity of the fractional derivative of a $T-$periodic function, the time-invariant fractional-order systems do not have any non-constant exact periodic solution unless the lower terminal of the derivative is $\pm\infty$ [12,13,23], which is not realistic. This property limits the applicability of the fractional derivative and makes it unfavorable for periodic real phenomena. In [1], the authors have proposed a modification of the Grünwald-Letnikov fractional differential operator, which consists of fixing the memory length and varying the lower terminal of the derivative. They have demonstrated that the modified definition of fractional derivative preserves the periodicity. The present paper extends this modification to the Caputo and Rieman-Liouville fractional-order operators. Tow examples are investigated to confirm that periodic solutions arise in fractional-order systems when the new fractional derivative operator is used. One linear system for which one gives an exact periodic solution defined by its analytical expression and another nonlinear system for which one provides an exact periodic solution using both qualitative and numerical methods.

2 Fractional-Order Derivatives

As said above, the most usual definitions of fractional-order derivative are the Grünwald-Letnikov, the Riemann-Liouville and the Caputo definitions [17]. For $0 < \alpha \notin \mathbb{N}$, the $\alpha-$th order derivative of a function $f(t)$ with respect to t and a terminal value a is given in the sense of

- Grünwald-Letnikov by

$$ {}_a^{GL}D_t^{\alpha} f(x) = \lim_{\substack{h \to 0 \\ nh = x-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x-kh), \tag{2.1} $$

where $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$.

- Riemann-Liouville by

$$ {}_a^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \Big(\int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau \Big). \tag{2.2} $$

- Caputo by

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{2.3} $$

In (2.2) and (2.3), m is the first integer greater than α, and $\Gamma(.)$ is the Gamma function. The following theorems reveal a remarkable property for the fractional derivatives based on Caputo definition, Grünwald-Letnikov definition, Riemann-Liouville definition [22].

Theorem 2.1. *Suppose that $f(t)$ is a non constant periodic function with period T. If $f(t)$ is m-times differentiable, then the functions ${}_{a}^{C}D_{t}^{\alpha}f(t)$, where $0 < \alpha \notin \mathbb{N}$ and m is the first integer greater than α, cannot be a periodic functions with period T.*

Theorem 2.2. *Suppose that $f(t)$ is (m-1)-times continuously differentiable and $f^{(m)}(t)$ is bounded. If $f(t)$ is a non-constant periodic function with period T, then the functions ${}_{a}^{GL}D_{t}^{\alpha}f(t)$ and ${}_{a}^{RL}D_{t}^{\alpha}f(t)$, where $0 < \alpha \notin \mathbb{N}$ and m is the first integer greater than α, cannot be periodic functions with period T.*

Example 2.3. Let $f(t) = \sin(t)$. One has

$$ \sin(t) = \sum_{p=0}^{\infty} (-1)^{p} \frac{t^{2p+1}}{(2p+1)!}. $$

Hence

$$ {}_{a}^{RL}D_{t}^{\alpha} \sin(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^{2}), $$

where $0 < \alpha < 1$ and $E_{\alpha,\beta}(t)$ is the generalized Mittag-Leffler function defined by

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k + \beta)}. $$

Numerical simulations showed that $t^{1-\alpha}E_{2,2-\alpha}(-t^{2})$ is not a periodic function where $0 < \alpha < 1$, even if $\alpha = 1$ this function is the periodic function $\cos(t)$.

As a consequence of the above theorems, periodic solution cannot be expected in fractional-order systems, under any circumstances [22,23].

Corollary 2.4. *A differential equation of fractional-order in the form*

$$ {}_{a}^{\cdot}D_{t}^{\alpha}x(t) = f(x(t)), $$

where $0 < \alpha \notin \mathbb{N}$, cannot have any non-constant smooth periodic solution.

This property highlights one of the basic differences between fractional-order derivative and integer-order one, and it makes fractional-order systems unfavourable for a wide range of real periodic phenomena. Therefore in this paper one overcomes this problem by imposing a simple modification to both Riemann-Liouville and Caputo definitions.

3 The Fractional-Order Derivative with Sliding Fixed Memory Length

one first recalls the Grünwald-Letnikov fractional-order derivative with fixed memory length introduced in [1].

Definition 3.1. (The Grünwald-Letnikov fractional derivative with fixed memory length) Let $\alpha \geq 0$, $L > 0$, m an integer such that $m-1 \leq \alpha < m$ and f an integrable function in the interval $[a-L, b]$. The operator ${}^{MG}_{\ L}D^{\alpha}_{t}$ defined by :

$$ {}^{MG}_{\ L}D^{\alpha}_{t}f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\frac{L}{h}}(-1)^{k}\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}f(t-kh), \quad t\in[a,b], \tag{3.1}$$

is called the Grünwald-Letnikov fractional derivative with sliding fixed memory length.

The following proposition gives an evaluation of the limit in the definition of Grünwald-Letnikov fractional derivative with sliding fixed memory length.

Proposition 3.2. *Under the assumptions of definition (3.1), if the function f is m-differentiable with $f^{(m)} \in L^1[a-L, b]$, then*

$$ {}^{MG}_{\ L}D^{\alpha}_{t}f(t) = \sum_{k=0}^{m-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)}\int_{t-L}^{t}(t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau. \tag{3.2}$$

It has been demonstrated that this modified fractional-order derivative possesses two important properties: the preservation of periodicity and the short memory, which considerably reduces the cost of numerical computations. Furthermore, it has been proven that contrarily to fractional autonomous systems defined using classical fractional derivative, the fractional autonomous systems in terms of the modified fractional derivative can generate exact periodic solutions.
In order to generalize this work, one introduces in this section a similar modification to both Caputo fractional-order derivative and Riemann-Liouville fractional-order derivative as follows.

Definition 3.3. (The Caputo fractional derivative with sliding fixed memory length) Let $\alpha > 0$, $L > 0$, m an integer such that $m = [\alpha] + 1$ and $f \in C^m[a-L, b]$. The Caputo fractional derivative with sliding fixed memory length is defined by

$$ {}^{MC}_{\ L}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t-L}^{t}(t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau. \tag{3.3}$$

Definition 3.4. (The Riemann-Liouville fractional derivative with sliding fixed memory length) Let $\alpha \geq 0$, $L > 0$, m an integer such that $m-1 \leq \alpha < m$ and f is a continuous function in $[a-L, b]$,the Riemann-Liouville fractional derivative with sliding fixed memory length is defined by

$$ {}^{MRL}_{\ \ L}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t-L}^{t}(t-\tau)^{m-\alpha-1}f(\tau)d\tau, \tag{3.4}$$

Remark 3.5. From (3.2) and (3.3) one gets

$$ {}^{MC}_{\ L}D^{\alpha}_{t}f(t) = {}^{MG}_{\ L}D^{\alpha}_{t}f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.5}$$

Proposition 3.6. *Under the assumption that the function $f(t)$ is m-times continuously differentiable*

$$ {}_{L}^{MRL}D_t^{\alpha} f(t) = {}_{L}^{MG}D_t^{\alpha} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.6} $$

Proof. By differentiation and performing repeatedly integration by parts, one has

$$ \begin{aligned} {}_{L}^{MRL}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \\ &= -\frac{f^{(m-1)}L^{m-\alpha-1}(t-L)}{\Gamma(m-\alpha)} + \frac{1}{\Gamma(m-\alpha-1)} \frac{d^{m-1}}{dt^{m-1}} \int_{t-L}^{t} (t-\tau)^{m-\alpha-2} f(\tau) d\tau, \\ &\ \vdots \\ &= -\sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau, \end{aligned} $$

setting $I = \frac{1}{\Gamma(-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha-1} f(\tau) d\tau$, and performing successive integrations by parts one obtains

$$ \begin{aligned} I &= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha} f'(\tau) d\tau, \\ &= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(t-L)L^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha+1} f^{(2)}(\tau) d\tau, \\ &\ \vdots \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha+m-1} f^{(m)}(\tau) d\tau, \\ &= {}_{L}^{MG}D_t^{\alpha} f(t). \end{aligned} $$

Therefore

$$ {}_{L}^{MRL}D_t^{\alpha} f(t) = {}_{L}^{MG}D_t^{\alpha} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

□

Remark 3.7. From (3.5) and (3.6) one has

$$ {}_{L}^{MRL}D_t^{\alpha} f(t) = {}_{L}^{MC}D_t^{\alpha} f(t) = {}_{L}^{MG}D_t^{\alpha} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.7} $$

In the following parts, one denotes the operators of Caputo and Riemann-Liouville fractional derivative with sliding fixed memory length by ${}_{L}^{M}D_t^{\alpha}$.

3.1 Fractional derivative of some elementary functions

In order to highlight the amazing properties of the fractional derivative with sliding fixed memory length one consider two elementary functions (the power and exponential functions), for which one computes their new derivatives.

3.1.1 New fractional derivative of the power function

Let $f(t) = t^n$, $n \in \mathbb{N}^*$, $\alpha > 0$, $L > 0$ and m is an integer such that $m - 1 < \alpha < m$.

If $n < m$, then $f^{(m)}(t) = 0$, substituting in (3.3) yields ${}^{M}_{L}D^{\alpha}_{t}(t^n) = 0$.
If $n \geq m$ then by repeated integrations by parts of the relation (3.3) one obtains

$$ {}^{M}_{L}D^{\alpha}_{t}(t^n) = \sum_{k=0}^{n-m} \frac{n! L^{-\alpha+m+k}(t-L)^{n-m-k}}{(n-m-k)!\Gamma(-\alpha+m+k+1)}. \tag{3.8} $$

Remark 3.8. (Fractional derivative of a constant function)
If f is a constant function (i.e.$f(t) = C$ for all $t \in [a - L, b]$, and C any constant including zero) then one has

$$ {}^{M}_{L}D^{\alpha}_{t}C = 0. $$

3.1.2 Fractional derivative of the exponential function

Let $f(t) = e^t = \sum\limits_{p=0}^{\infty} \frac{t^p}{p!}$, $\alpha > 0$, $L > 0$ and m is an integer such that $m - 1 < \alpha < m$.

One has

$$ {}^{M}_{L}D^{\alpha}_{t}e^t = {}^{M}_{L}D^{\alpha}_{t} \sum_{p=0}^{\infty} \frac{t^p}{p!} = \sum_{p=0}^{\infty} \frac{1}{p!} \, {}^{M}_{L}D^{\alpha}_{t}t^p. $$

From (3.8), one obtains that

$$ \begin{aligned} {}^{M}_{L}D^{\alpha}_{t}(e^t) &= \sum_{p=0}^{\infty} \sum_{k=0}^{p-m} \frac{L^{-\alpha+m+k}(t-L)^{p-m-k}}{(p-m-k)!\Gamma(-\alpha+m+1+k)}, \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^{p-m} \frac{L^{-\alpha+m+k}(t-L)^{p-m-k}}{(p-m-k)!\Gamma(k-\alpha+m+1)}, \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{L^{-\alpha+m+k}(t-L)^{p-k}}{(p-k)!\Gamma(k-\alpha+m+1)}, \\ &= \sum_{p=0}^{\infty} \frac{L^{-\alpha+m}(t-L)^p}{p!\Gamma(-\alpha+m+1)} + \sum_{p=0}^{\infty} \frac{L^{-\alpha+m+1}(t-L)^p}{p!\Gamma(-\alpha+m+2)} + \ldots, \\ &= \left(\sum_{p=0}^{\infty} \frac{(t-L)^p}{p!\Gamma(-\alpha+m+1)} \right) \left(\sum_{k=0}^{\infty} \frac{L^{-\alpha+m+k}}{\Gamma(-\alpha+m+1+k)} \right), \\ &= e^{t-L} L^{-\alpha+m} \sum_{k=0}^{\infty} \frac{L^k}{\Gamma(-\alpha+m+1+k)}, \\ &= e^{t-L} L^{m-\alpha} E_{1,m+1-\alpha}(L). \end{aligned} $$

3.2 Derivative of a periodic function

The main result of this paper is stated in the following theorem.

Theorem 3.9. *Let $\alpha > 0$, $L > 0$ and m an integer such that $m - 1 < \alpha < m$ and $f \in C^m[a - L, b]$. If f is a periodic function with period T, Then ${}^{M}_{L}D^{\alpha}_{t}f$ is a periodic function with the same period T.*

Proof. Suppose that f is a periodic function with a period T. The aim of this proof is to demonstrate that the function $g(t) = {}^{M}_{L}D^{\alpha}_{t}f$ is a periodic function with the same period T (i.e.

$g(t+T) = g(t)$).
One has

$$
\begin{aligned}
g(t+T) = {}_{L}^{M}D_{t+T}^{\alpha} f(t+T) &= \frac{1}{\Gamma(m-\alpha)} \int_{t+T-L}^{t+T} (t+T-\tau)^{m-\alpha-1} f^{(m)}(\tau+T) d\tau, \\
&= \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s+2T) ds, \\
&= \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \\
&= {}_{L}^{M}D_{t}^{\alpha} f(t) = g(t).
\end{aligned}
$$

Thus, ${}_{L}^{M}D_{t}^{\alpha} f$ is a periodic function with the same period T.

□

3.2.1 Fractional derivative of some fundamental periodic functions

Note first that the functions ${}_{L}^{MG}D_{t}^{\alpha} \sin(t)$ and ${}_{L}^{MG}D_{t}^{\alpha} \cos(t)$ have been calculated in [1].

Example 3.10. (Fractional derivative with sliding fixed memory length of the sine function)
By definition

$$
{}_{L}^{M}D_{t}^{\alpha} f(t) = {}_{L}^{MG}D_{t}^{\alpha} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L) L^{k-\alpha}}{\Gamma(k-\alpha+1)}.
$$

Therefore

$$
\begin{aligned}
{}_{L}^{M}D_{t}^{\alpha} \sin(t) = {}_{L}^{MG}D_{t}^{\alpha} \sin(t) &- \sum_{k=0}^{m-1} \frac{\frac{d^k}{dt^k}(\sin(t-L)) L^{k-\alpha}}{\Gamma(k-\alpha+1)}, \\
&= L^{-\alpha} \sin(t-L) E_{2,1-\alpha}(-L^2) + L^{1-\alpha} \cos(t-L) E_{2,2-\alpha}(-L^2) \\
&- L^{-\alpha} \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)} \sin(t-L) - L^{1-\alpha} \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)} \cos(t-L), \\
&= L^{-\alpha} \sin(t-L) (E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}) \\
&+ L^{1-\alpha} \cos(t-L) (E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}), \\
&= a \sin(t-L) + b \cos(t-L),
\end{aligned}
\tag{3.9}
$$

where, $a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)})$, $b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)})$.
One notices that, ${}_{L}^{M}D_{t}^{\alpha} \sin(t)$ is a periodic function with the period $T = 2\pi$. This analytical result is displayed in figure (3.1), for some values of α and $L = 32.1$.

Example 3.11. (Fractional derivative of cosine function)

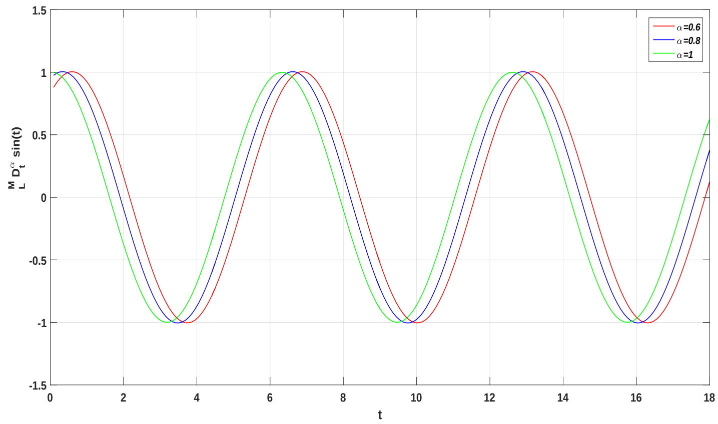

Figure 3.1: Fractional derivative of the Sine function for $L = 32.1$ and some values of α.

By definition

$$
\begin{aligned}
{}^{M}_{L}D^{\alpha}_{t}\cos(t) &= {}^{MG}_{\ L}D^{\alpha}_{t}\cos(t) - \sum_{k=0}^{m-1}\frac{\frac{d^k}{dt^k}(\cos(t-L))L^{k-\alpha}}{\Gamma(k-\alpha+1)},\\
&= L^{-\alpha}\cos(t-L)E_{2,1-\alpha}(-L^2) - L^{1-\alpha}\sin(t-L)E_{2,2-\alpha}(-L^2)\\
&- L^{-\alpha}\sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}\cos(t-L) + L^{1-\alpha}\sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}\sin(t-L),\\
&= L^{-\alpha}\cos(t-L)(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)})\\
&- L^{1-\alpha}\sin(t-L)(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}),\\
&= a\cos(t-L) - b\sin(t-L),
\end{aligned}
\tag{3.10}
$$

where,

$$a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}),$$

and

$$b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}).$$

Obviously ${}^{M}_{L}D^{\alpha}_{t}\cos(t)$ is a periodic function with period $T = 2\pi$.

3.3 An interpolation property

It is known that the operator of Grünwald-Letnikov fractional derivative with sliding fixed memory length is an extension of the integer-order operator $\frac{d^m}{t^m}$, (see [1]).
The following proposition proves that the Caputo and Riemann-Liouville operators of the

fractional derivative with sliding fixed memory length verifies this property for $\alpha \to m$, but not for $\alpha \to m-1$.

Proposition 3.12. *Let $L > 0$ and $0 \le m-1 < \alpha < m$ such that m is an integer number, and let $f(t)$ having $(m+1)$ continuous bounded derivatives in $[a-L, b]$. Then , for all $t \in [a, b]$, one has*

$$\lim_{\alpha \to m} {}_{L}^{M}D_t^{\alpha} f(t) = f^{(m)}(t),$$

and

$$\lim_{\alpha \to m-1} {}_{L}^{M}D_t^{\alpha} f(t) = f^{(m-1)}(t) - f^{(m-1)}(t-L).$$

Proof. One has

$$\begin{aligned}\lim_{\alpha \to m} {}_{L}^{M}D_t^{\alpha} f(t) &= \lim_{\alpha \to m} \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,\\ &= \lim_{\alpha \to m} \frac{L^{m-\alpha} f^{(m)}(t-L)}{\Gamma(m-\alpha+1)} + \lim_{\alpha \to m} \frac{1}{\Gamma(m-\alpha+1)}\\ &\int_{t-L}^{t} (t-\tau)^{m-\alpha} f^{(m+1)}(\tau) d\tau,\\ &= f^{(m)}(t-L) + \int_{t-L}^{t} f^{(m+1)}(\tau) d\tau,\\ &= f^{(m)}(t).\end{aligned}$$

For $\alpha \to m-1$, one has

$$\begin{aligned}\lim_{\alpha \to m-1} {}_{L}^{MC}D_t^{\alpha} f(t) &= \lim_{\alpha \to m-1} \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,\\ &= \int_{t-L}^{t} f^{(m)}(\tau) d\tau,\\ &= f^{(m-1)}(t) - f^{(m-1)}(t-L).\end{aligned}$$

□

Example 3.13.
Let $f(t) = e^t$, then

$${}_{L}^{M}D_t^{\alpha} e^t = e^{t-L} L^{m-\alpha} E_{1,m+1-\alpha}(L),$$

Therefore,

$$\lim_{\alpha \to m} {}_{L}^{M}D_t^{\alpha} e^t = e^{t-L} E_{1,1}(L) = e^t = f^{(m)}(t).$$

However,

$$\begin{aligned}\lim_{\alpha \to m-1} {}_{L}^{M}D_t^{\alpha} e^t &= e^{t-L} L E_{1,2}(L) = e^{t-L}(e^L - 1),\\ &= e^t - e^{t-L} = f^{(m)}(t) - f^{(m-1)}(t-L).\end{aligned}$$

Example 3.14.
Let $f(t) = t^n$, one has

$${}_{L}^{M}D_t^{\alpha}(t^n) = \sum_{k=0}^{n-m} \frac{n! L^{-\alpha+m+k} (t-L)^{n-m-k}}{(n-m-k)! \Gamma(-\alpha+m+k+1)}.$$

Putting $N = n - m$ and $t - L = a$, then

$$\begin{aligned}
\lim_{\alpha \to m} {}^{M}_{L}D^{\alpha}_{t}(t^n) &= \sum_{k=0}^{N} \frac{n! L^k a^{N-k}}{(N-k)!k!}, \\
&= \frac{n!}{N!} \sum_{k=0}^{N} \frac{N! L^k a^{N-k}}{(N-k)!k!}, \\
&= \frac{n!}{N!}(a+L)^N = \frac{n!}{(n-m)!} t^{n-m}, \\
&= \frac{d^m}{dt} t^n = f^{(m)}(t).
\end{aligned}$$

However,

$$\begin{aligned}
\lim_{\alpha \to m-1} {}^{M}_{L}D^{\alpha}_{t}(t^n) &= \sum_{k=0}^{N} \frac{n! L^{k+1} a^{N-k}}{(N-k)!(k+1)!}, \\
&= \frac{n!}{(N+1)!} \sum_{k=0}^{N+1} \frac{(N+1)! L^k a^{N+1-k}}{(N+1-k)!k!} - \frac{n!}{(n-m+1)!}(t-L)^{n-m+1}, \\
&= \frac{n!}{(N+1)!} t^{N+1} - \frac{n!}{(n-(m-1))!}(t-L)^{n-(m-1)}, \\
&= \frac{n!}{(n-(m-1))!} t^{n-(m-1)} - \frac{n!}{(n-(m-1))!}(t-L)^{n-(m-1)}, \\
&= \frac{d^{m-1}}{dt} t^n - \frac{d^{m-1}}{dt}(t-L)^n = f^{(m-1)}(t) - f^{(m-1)}(t-L).
\end{aligned}$$

3.4 Comparison between some results of classical fractional-order derivatives and fractional order derivatives with sliding fixed memory length

The previous results are summarized in the table (3.1), in order to highlight the differences between classical fractional-order derivative and fractional-order derivative with sliding fixed memory length.

3.5 Fractional-order autonomous system with exact periodic solution

As previously mentioned, any autonomous fractional-order system expressed in terms of classical fractional derivatives cannot have any exact periodic solutions [13,22,23].
Conversely to these results, one presents some examples showing that fractional-order autonomous systems (linear and nonlinear) expressed in terms of fractional derivatives with sliding fixed memory length can have exact periodic solutions.

Example 3.15. (Linear fractional-order system)
Let consider the following linear fractional-order autonomous system

$$ {}^{M}_{L}D^{\alpha}_{t}X(t) = AX(t), \tag{3.11}$$

where $X(t) \in \mathbb{R}$ and $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, with $a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{p=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^p}{\Gamma(2p+1-\alpha)})$,
$b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{p=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^p}{\Gamma(2p+2-\alpha)})$.

Classical fractional derivative ${}^{C}_{a}D^{\alpha}_{t}$ or ${}^{RL}_{a}D^{\alpha}_{t}$	**Fractional derivative with sliding fixed memory length** ${}^{M}_{L}D^{\alpha}_{t}$
${}^{C}_{a}D^{\alpha}_{t}f(t) = {}^{RL}_{a}D^{\alpha}_{t}f(t) - \sum\limits_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}$	${}^{MC}_{L}D^{\alpha}_{t}f(t) = {}^{MR}_{L}D^{\alpha}_{t}f(t)$
$\lim\limits_{\alpha\to m} {}^{RL}_{a}D^{\alpha}_{t}f(t) = \lim\limits_{\alpha\to m} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(m)}(t)$	$\lim\limits_{\alpha\to m} {}^{M}_{L}D^{\alpha}_{t}f(t) = f^{(m)}(t)$
$\lim\limits_{\alpha\to m-1} {}^{RL}_{a}D^{\alpha}_{t}f(t) = f^{(m-1)}(t)$, $\lim\limits_{\alpha\to m-1} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(m-1)}(t) - f^{(m-1)}(a)$	$\lim\limits_{\alpha\to m-1} {}^{M}_{L}D^{\alpha}_{t}f(t) = f^{(m-1)}(t) - f^{(m-1)}(t-L)$
${}^{RL}_{0}D^{\alpha}_{t}(t^n) = {}^{C}_{0}D^{\alpha}_{t}(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}$	${}^{M}_{L}D^{\alpha}_{t}(t^n) = \sum\limits_{k=0}^{n-m}\frac{n!L^{-\alpha+m+k}(t-L)^{n-m-k}}{(n-m-k)!\Gamma(-\alpha+m+k+1)}$
${}^{RL}_{a}D^{\alpha}_{t}C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{\alpha} \neq 0$, ${}^{C}_{a}D^{\alpha}_{t}C = 0$	${}^{M}_{L}D^{\alpha}_{t}C = 0$
${}^{RL}_{a}D^{\alpha}_{t}\sin t = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$	${}^{M}_{a}D^{\alpha}_{t}\sin t = a\sin(t-L) + b\cos(t-L)$.

Table 3.1: Comparison between some results of classical fractional-order derivatives and fractional order derivatives with sliding fixed memory length.

- For $L = 2k\pi$, where k is a non-zero integer. The vector function $X(t) = c\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$, $c \in \mathbb{R}$ is an exact $2\pi-$periodic solution for the system (3.11).
By definition,

$$ {}^{M}_{2k\pi}D^{\alpha}_{t}X(t) = c\begin{pmatrix} {}^{M}_{2k\pi}D^{\alpha}_{t}\cos(t)\\ {}^{M}_{2k\pi}D^{\alpha}_{t}\sin(t)\end{pmatrix}. $$

Then, from (3.9) and (3.10) one obtains

$$ \begin{aligned} {}^{M}_{2k\pi}D^{\alpha}_{t}X(t) &= c\begin{pmatrix} a\cos(t-2k\pi) - b\sin(t-2k\pi)\\ a\sin(t-2k\pi) + b\cos(t-2k\pi)\end{pmatrix},\\ &= c\begin{pmatrix} a & -b\\ b & a\end{pmatrix}\begin{pmatrix}\cos(t-2k\pi)\\ \sin(t-2k\pi)\end{pmatrix},\\ &= cA\begin{pmatrix}\cos(t-2k\pi)\\ \sin(t-2k\pi)\end{pmatrix},\\ &= AX(t). \end{aligned} $$

Therefore, $X(t) = c\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$ is an exact $2\pi-$periodic solution of (3.11) with $L = 2k\pi$.
- For $L = \frac{\pi}{2}$, one has

$$ \begin{aligned} {}^{M}_{\frac{\pi}{2}}D^{\alpha}_{t}X(t) &= c\begin{pmatrix} a\cos(t-\frac{\pi}{2}) - b\sin(t-\frac{\pi}{2})\\ a\sin(t-\frac{\pi}{2}) + b\cos(t-\frac{\pi}{2})\end{pmatrix},\\ &= c\begin{pmatrix} a\sin(t) + b\cos(t)\\ -a\cos(t) + b\sin(t)\end{pmatrix},\\ &= c\begin{pmatrix} b & a\\ -a & b\end{pmatrix}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix},\\ &= cB\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix},\\ &= BX(t), \end{aligned} $$

with $B = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \neq A$. Thus, $X(t) = c \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ is not solution of (3.11), but it is an exact $2\pi-$periodic solution of the system ${}^{M}_{\frac{\pi}{2}}D^{\alpha}_{t}X(t) = BX(t)$.

Example 3.16. (The predator-prey model with Holling type II response function)
All population species possess the property of heredity, which means the passing on traits from parents to their offspring, either through asexual reproduction or sexual reproduction. The offspring cells or organisms acquire the genetic information of their parents through heredity. This property makes fractional differential systems models more efficiently regarding some specific problems than ordinary differential ones.
Motivated by this fact, we introduce the fractional version of the Holling-Tanner model [21] as follows

$$\begin{cases} D^{\alpha}x = r_1 x(1 - \frac{x}{K}) - \frac{qxy}{m+x}, \\ D^{\alpha}y = r_2 y(1 - \frac{y}{\gamma x}). \end{cases} \tag{3.12}$$

Where $D^{\cdot}$ denotes a standard fractional-order derivative operator and $\alpha \in [0, 1]$ is the fractional-order related to the hereditary property of the population (a value of α close to an integer number means that the population has a weak hereditary property), $x(t) \geq 0$ and $y(t) \geq 0$ are the density of prey and predator populations at time t respectively. The parameters r_1 and r_2 are the intrinsic growth rates, K represents the carrying capacity of the prey, q is the maximum number of preys that can be eaten per predator per unit of time, m is the saturation value (it corresponds to the number of preys necessary to achieve one half the maximum rate q), γ is a measure of the quality of the prey as a portion of food for the predator.

Since exact analytical resolution of this nonlinear system is unavailable, one resorts to qualitative and numerical study. For this purpose the parameters are set to $r_1 = 1$, $r_2 = 0.2$, $K = 25$, $q = \frac{6}{7}$, $m = 1$ and $\gamma = 0.95$, the system (3.12) has two equilibrium points $E_0 = (25, 0)$ and $E_1 \approx (7.1429, 6.7857)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_0 is given by

  $$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = \lambda^2 + 0.8\lambda - 0.2.$$

  So $a_2 = -0.2 < 0$, then according to Proposition 1 in [7] E_0 is unstable for all $\alpha \in [0, 2)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_1 is given by

  $$P(\lambda) = \lambda^2 - 0.1409\lambda + 0.0747.$$

  So $a_1 \approx -0.1409$ and $a_2 \approx 0.0747 > 0$.

  Applying Hopf-Like Bifurcation theory [2–4] and using Proposition 1 in [7], one obtains the Hopf-Like bifurcation value

  $$\alpha^* = \frac{2}{\pi} cos^{-1}(\frac{-a_1}{2\sqrt{a_2}}) \approx 0.8341.$$

  The fixed point E_1 losses its stability, and a periodic motion ($S-$asymptotically periodic for the classical fractional derivative and exact periodic for fractional derivative with sliding fixed memory length) appears.

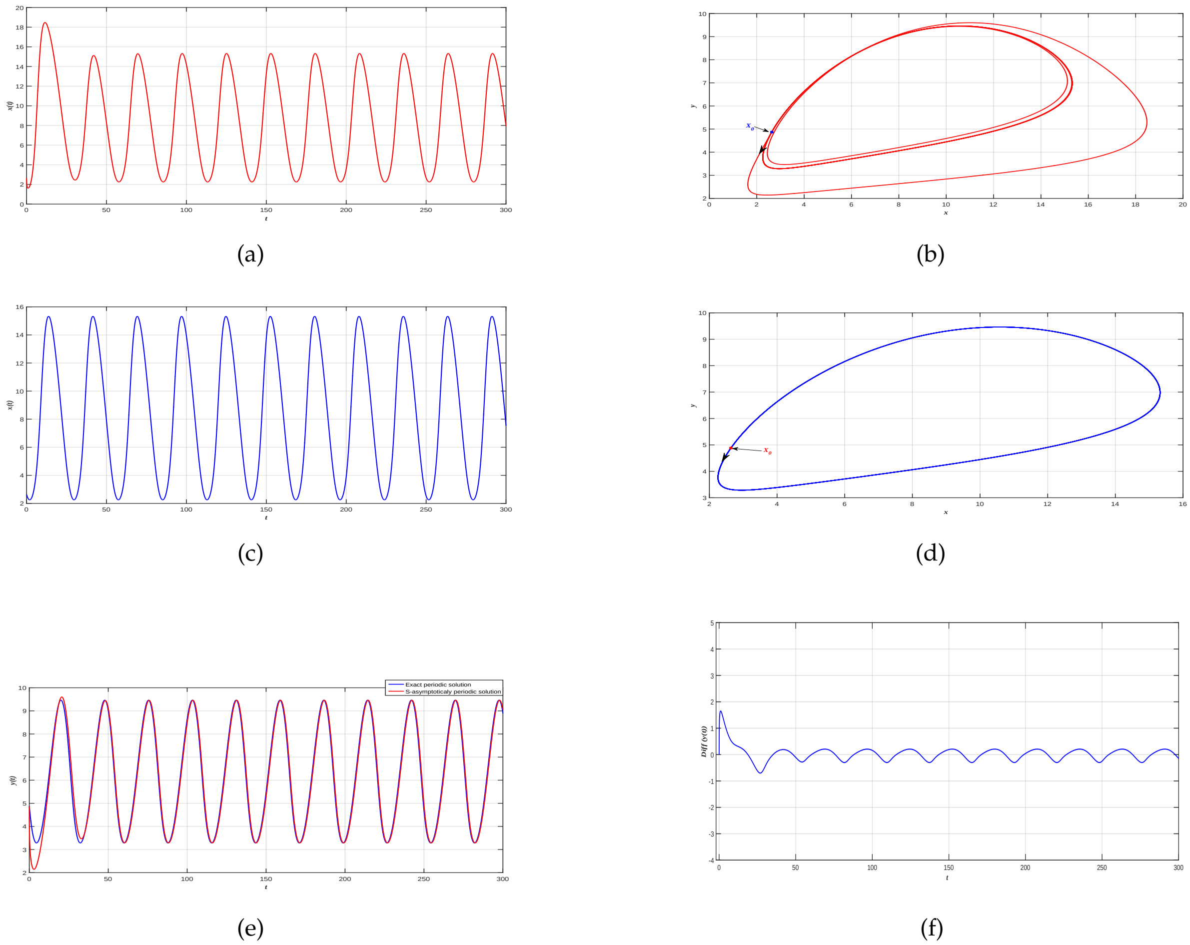

Figure 3.2: Time evolution and phase portrait of system (3.12) for $\alpha = 0.9$ (a,b) $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ for classical fractional operator. (c,d) Exact $T-$periodic solution for the fractional derivative operator with sliding fixed memory length. (e,f) Comparison between the two solutions.

To illustrate these results, one solves the system (3.12) numerically by developing a Matlab code using a discretization technique based on the formula (3.7).
Choosing a value for α greater than α^*, for example, $\alpha = 0.9$, one compares the solution of (3.12) in terms of classical fractional operator and its solution in terms of the fractional operator with sliding fixed memory length $L = 30$. The two trajectories are start from the same initial point $X_0 = (2.64, 4.88)$, belonging to the attracting limit cycle. The results are shown in Fig. 3.2.
An $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ is obtained for classical fractional operator as shown in Fig. 3.2(a,b), and an exact $T-$periodic solution is obtained for the fractional derivative operator with sliding fixed memory length as shown in Fig. 3.2(c,d).

4 Conclusion

In this article, one modifies the Caputo and Rieman-Liouville fractional-order derivatives by fixing the memory length and varying the lower terminal of the derivative. It is shown that

the modified fractional derivative operator preserves the periodicity. Consequently, periodic solutions can be obtained in fractional-order systems expressed in terms of the new operator. Two examples are investigated to highlight this property for a linear system provides an analytic expression of an exact periodic solution is computed and for another nonlinear system for which exact periodic solutions are obtained using qualitative and numerical methods.

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Conflict of Interest

The authors have no conflicts of interest to disclose.

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